Euler-Heisenberg lagrangians and asymptotic analysis in 1+1 QED, part 1: Two-loop

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Abstract: We continue an effort to obtain information on the QED perturbation series at high loop orders, and particularly on the issue of large cancellations inside gauge invariant classes of graphs, using the example of the l - loop N - photon amplitudes in the limit of large photons numbers and low photon energies. As was previously shown, high-order information on these amplitudes can be obtained from a nonperturbative formula, due to Affleck et al., for the imaginary part of the QED effective lagrangian in a constant field. The procedure uses Borel analysis and leads, under some plausible assumptions, to a number of nontrivial predictions already at the three-loop level. Their direct verification would require a calculation of this 'Euler-Heisenberg lagrangian' at three-loops, which seems presently out of reach. Motivated by previous work by Dunne and Krasnansky on Euler-Heisenberg lagrangians in various dimensions, in the present work we initiate a new line of attack on this problem by deriving and proving the analogous predictions in the simpler setting of 1+1 dimensional QED. In the first part of this series, we obtain a generalization of the formula of Affleck et al. to this case, and show that, for both Scalar and Spinor QED, it correctly predicts the leading asymptotic behaviour of the weak field expansion coefficients of the two loop Euler-Heisenberg lagrangians.

1 Introduction

QED is the oldest and prototypical quantum field theory, and much effort has gone into calculating low order terms in its perturbation series. However, little is still known about its high-order behaviour, in fact less than for some other field theories [1, 2, 3, 4, 5]. This is due, on one hand, to the absence of (spacetime) instantons in the abelian case, on the other hand to large-scale cancellations between Feynman diagrams which are familiar to all practicioners in the field, but whose origin and effect on the large order behaviour of perturbation theory is still not well-understood. This type of cancellations has recently attracted wider attention since it is now increasingly believed that their study may throw light on the origin of similar cancellations observed in gravity and supergravity theories (see, e.g., [6, 7] and refs. therein).

In their pioneering calculation of the $g-2$ factor of the electron to sixth order in 1974, Cvitanovic and Kinoshita [8] found a coefficient which was much smaller numerically than had been expected by a naive estimate based on the number of Feynman diagrams involved. A detailed analysis revealed extensive cancellations inside gauge invariant classes of diagrams. This led Cvitanovic [9] to conjecture that, at least in the quenched approximation (i.e. excluding diagrams involving virtual fermions) these cancellations would be important enough numerically to render this series convergent for the $g-2$ factor (see [10] for an amusing account of the genesis of this conjecture). Although nowadays there exist many good arguments against convergence of the QED perturbation series, all of them are based on the presence of an unlimited number of virtual fermions, so that Cvitanovic's conjecture for the quenched contribution is still open today. Moreover, should it hold true for the case of the $g-2$ factor, it is natural to assume that it extends to arbitrary QED amplitudes, such as the quenched photon S-matrix (for photon amplitudes, we call 'quenched' the contributions involving only one scalar/fermion loop). Beyond the quenched approximation, it would suggest that the QED perturbation series should be rearranged as a series in the number of fermion loops, rather than in the coupling constant.

The present work continues an effort [11, 12, 13, 14] to study the multi-loop behaviour of the QED N photon amplitudes using the QED effective lagrangian, and in particular to prove or disprove Cvitanovic's conjecture for these amplitudes. Let us start with recalling the representation obtained by Heisenberg and Euler [15] for the one-loop QED effective Lagrangian in a constant field,

$$\mathcal{L}^{(1)}_{\rm spin}(F) = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,{\rm e}^{-m^2T}\biggl[\frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{1}{3}(a^2-b^2)T^2 - 1\biggr] \tag{1.1}$$

Here T is the proper-time of the loop particle and a, b are defined by $a^2-b^2 = B^2-E^2, \quad ab = \mathbf{E}\cdot\mathbf{B}$. The Euler-Heisenberg Lagrangian ('EHL' in the following) contains the information on the one-loop N photon amplitudes in the limit where all photon energies are small compared to the electron mass (see, e.g., [16, 17]). After expanding the EHL in powers of the field invariants, the 'weak field expansion', it is straightforward to obtain the explicit form of the N photon amplitudes in this 'EH limit' from the terms in this expansion involving N powers of the field. Using the spinor helicity formalism, the result of this procedure can be expressed quite concisely [13]:

$$\begin{aligned}\Gamma^{(EH)}_{\rm spin}[\varepsilon_1^+;\ldots;\varepsilon_K^+;\varepsilon_{K+1}^-;\ldots;\varepsilon_N^-] &= -\frac{m^4}{8\pi^2}\Bigl(\frac{2ie}{m^2}\Bigr)^N (N-3)!\\ &\times\sum_{k=0}^{K}\sum_{l=0}^{N-K}(-1)^{N-K-l}\frac{B_{k+l}B_{N-k-l}}{k!l!(K-k)!(N-K-l)!}\chi_K^+\chi_{N-K}^-\end{aligned} \tag{1.2}$$

Here the superscripts $\pm$ refer to circular polarizations, and the B_k are Bernoulli numbers. The invariants $\chi_K^\pm$ are written, in standard spinor helicity notation (see, e.g., [18]),

$$\begin{aligned}\chi_K^+ &= \frac{(\frac{K}{2})!}{2^{\frac{K}{2}}}\Bigl\{[12]^2[34]^2\cdots[(K-1)K]^2 + \text{ all permutations}\Bigr\}\\ \chi_{N-K}^- &= \frac{(\frac{N-K}{2})!}{2^{\frac{N-K}{2}}}\Bigl\{\langle(K+1)(K+2)\rangle^2\langle(K+3)(K+4)\rangle^2\cdots\langle(N-1)N\rangle^2\\ &\qquad + \text{all perm.}\Bigr\}\end{aligned} \tag{1.3}$$

The formula analogous to (1.1) for Scalar QED is

$$\mathcal{L}^{(1)}_{\rm scal}(F) = \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\,{\rm e}^{-m^2T}\biggl[\frac{(eaT)(ebT)}{\sinh(eaT)\sin(ebT)} + \frac{1}{6}(a^2-b^2)T^2 - 1\biggr] \tag{1.4}$$

This formula was obtained by Weisskopf [19], but for simplicity will be called "scalar EHL" in the following.

Except for the purely magnetic case, the effective Lagrangians (1.1),(1.4) have an imaginary part. For the purely electric case, Schwinger [20] found the following representation of the imaginary parts in terms of infinite series of 'Schwinger exponentials',

$$\begin{aligned} \mathrm{Im}\mathcal{L}^{(1)}_{\rm scal}(E) &= -\frac{(eE)^2}{16\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\exp\left[-\pi k\frac{m^2}{eE}\right] \qquad (1.5)\\ \mathrm{Im}\mathcal{L}^{(1)}_{\rm spin}(E) &= \frac{(eE)^2}{8\pi^3}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left[-\pi k\frac{m^2}{eE}\right] \qquad (1.6) \end{aligned}$$

These formulas imply that any constant electric field will lead to a certain probability for electron-positron pair creation from vacuum. The inverse exponential dependence on the field suggests to think of this as a tunneling process in which virtual pairs draw enough energy from the field to turn real, as had been proposed by Sauter as early as 1931 [21].

At the one-loop level, obtaining the imaginary part of the EHL from its real part is a simple application of Cauchy's theorem. For our multiloop purposes, however, it will be important that the imaginary part can, using Borel summation, also be obtained from the asymptotic behaviour of the coefficients of the weak-field expansion. Referring to [5, 22] for the details of this procedure, let us just cite the following main result: Assume that a function $f(g)$ has an asymptotic series expansion

$$f(g) \sim \sum_{n=0}^{\infty} c_n g^n \qquad (1.7)$$

where the expansion coefficients c_n have the leading-order large n behaviour

$$c_n \sim \rho^n \Gamma(\mu n + \nu) \qquad (1.8)$$

with some real constants $\rho > 0$, $\mu > 0$ and ν. Then the series (1.7) has convergence radius zero, and is not even Borel-summable. Nevertheless, one can, by a dispersion relation applied to the formal Borel integral, show that the leading imaginary contribution for small g is given by

$$\mathrm{Im} f(g) \sim \frac{\pi}{\mu}\left(\frac{1}{\rho g}\right)^{\nu/\mu} \exp\left[-\left(\frac{1}{\rho g}\right)^{1/\mu}\right] \tag{1.9}$$

These formulas can be applied to the weak field expansions of the purely electric EHL's (1.1), (1.4), setting $g = (eE/m^2)^2$. The expansion coefficients c_n involve essentially only the Bernoulli numbers [17, 22], and using the asymptotic properties of those numbers it is easy to see that they obey (1.8) with $\rho = 1/\pi^2$, $\mu = 2$, and $\nu = -2$, independently of spin. Eq. (1.9) then reproduces the $k = 1$ terms in (1.5) and (1.6). One can then go on and iterate this procedure to successively construct all the Schwinger exponentials [22]. However, in the following we will generally be concerned only with the leading term.

Considerable work has gone into generalizing the EHL's to the two loop level, that is, taking a single photon exchange in the loop into account. For a general constant field, various equivalent integral representations have been found for these two-loop EHL's $\mathcal{L}^{(2)}_{\mathrm{spin}}(F)$ [23, 24, 25, 26, 27, 28, 29] and $\mathcal{L}^{(2)}_{\mathrm{scal}}(F)$ [30, 24, 26, 27, 28, 29], all at the two-parameter integral level. So far these integrals have withstood attempts at analytical evaluation. However, they have been used to compute the two-loop weak field expansion coefficients to high orders [26, 27, 22], and to derive closed-form expressions for these coefficients in the purely electric/magnetic case [31].

The Schwinger representations (1.5),(1.6) also generalize to the two-loop level [32, 24], although now with a certain prefactor function $K_k(eE/m^2)$ in front of the kth Schwinger exponential; those functions are known explicitly only at leading orders in the weak-field limit.

However, it turns out that the electric or magnetic backgrounds are not the simplest ones in this context. Computationally, the most favorable case is the one of a (euclidean) self-dual ('SD') field, defined by $F = \tilde{F}$, which has the consequence that

$$F^2 = -f^2 \mathbb{1} \tag{1.10}$$

For a real value of the parameter f, the SD effective Lagrangian has properties similar to the magnetic EHL, for imaginary f similar to the electric one. In this SD case, even at two loops it is possible to obtain explicit formulas for the EHL's [11, 12]:

$$\mathcal{L}^{(2)(SD)}_{\rm scal}(\kappa) = \alpha \frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\Bigl[\frac{3}{2}\xi^2(\kappa) - \xi'(\kappa)\Bigr] \tag{1.11}$$

$$\mathcal{L}^{(2)(SD)}_{\rm spin}(\kappa) = -2\alpha \frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\bigl[3\xi^2(\kappa) - \xi'(\kappa)\bigr] \tag{1.12}$$

Here we have defined the convenient dimensionless parameter

$$\kappa \equiv \frac{m^2}{2ef} \tag{1.13}$$

as well as the function

$$\xi(x) \equiv -x\Bigl(\psi(x) - \ln(x) + \frac{1}{2x}\Bigr) \tag{1.14}$$

where ψ is the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$. Thus in the self-dual case the study of the weak-field expansions and the construction of the imaginary parts involve only some well-known properties of the digamma function, making it possible to do everything much more completely and explicitly than for the electric or magnetic case [12]. In particular, for this case it has been verified that certain complications related to the non-uniqueness of f, which could invalidate the relation between (1.8) and (1.9), such as the appearance of additional poles or cuts in the complex g plane, do not occur in QED, or at least not yet at the two-loop level.

Moreover, although a self-dual field cannot be realized in Minkowski space, the corresponding EHL still contains information on the N – photon amplitudes, namely on their "all +" component in the helicity decomposition (1.2) [33].

Beyond the two-loop level, to the best of our knowledge the only result on EHL's in the literature is the following formula, proposed in 1982 by Affleck, Alvarez, and Manton [34] as an all-loop generalization of the leading Schwinger exponential for scalar QED (1.5):

$$\mathrm{Im}\mathcal{L}^{(\mathrm{all-loop})}_{\rm scal}(E) \overset{E\to 0}{\sim} \frac{(eE)^2}{16\pi^3}\exp\Bigl[-\pi\frac{m^2}{eE} + \alpha\pi\Bigr] \tag{1.15}$$

This formula (called 'AAM formula' in the following) is highly remarkable for various reasons. Despite of its simplicity it is a true all-loop result; the rhs receives contributions from an infinite set of Feynman diagrams of arbitrary loop order, as sketched in fig. 1.

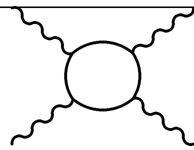
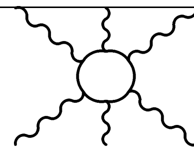
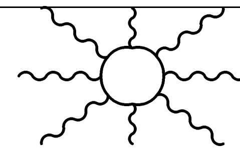
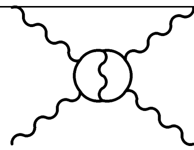
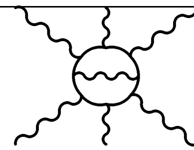
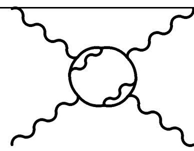
	Number of external legs			
Number of loops	4	6	8	$\cdots$
1				$\cdots$
2			$\cdots$	$\cdots$
3		$\cdots$	$\ddots$	$\vdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$

Figure 1: Diagrams contributing to $\mathrm{Im}\mathcal{L}_{\mathrm{scal}}^{(\mathrm{all-loop})}(E)$ in the weak-field limit.

Moreover, the mass appearing in (1.15) is argued to be still the physical renormalized mass, which means that the above figure should strictly speaking include also the mass renormalization counter diagrams which appear in EHL calculations starting from two loops.

The derivation given in [34] is very simple, if formal. Based on a stationary path approximation of Feynman's worldline path integral representation [35] of $\mathcal{L}_{\mathrm{scal}}(E)$, it actually uses only a one-loop semiclassical trajectory, and arguments that this trajectory remains valid in the presence of virtual photon insertions. This also implies that non-quenched diagrams do not contribute in the limit (1.15), which is why we have shown only the quenched ones in fig. 1.

Although the derivation of (1.15) in [34] cannot be considered rigorous, an independent heuristic derivation of (1.15), as well as extension to the spinor QED case (with the same factor of $\mathrm{e}^{\alpha\pi}$) was given by Lebedev and Ritus [32] through the consideration of higher-order corrections to the pair creation energy in the vacuum tunneling picture. At the two-loop level, (1.15) and its spinor QED extension state that

$$\mathrm{Im}\mathcal{L}^{(2)}_{\rm scal,spin}(E) \quad \overset{E\to 0}{\sim} \quad \alpha\pi\, \mathrm{Im}\mathcal{L}^{(1)}_{\rm scal,spin}(E) \tag{1.16}$$

which has been verified by a direct calculation of the EHL [22] (for the spinor QED case), and also been extended to the case of a self-dual field [11, 12].

Now, writing the AAM formula (1.15) as (hereafter, unless indicated otherwise, we refer to the scalar and spinor QED cases simultaneously)

$$\mathrm{Im}\mathcal{L}^{(\rm all-loop)}(E) \;=\; \sum_{l=1}^{\infty} \mathrm{Im}\mathcal{L}^{(l)}(E) \quad \overset{E\to 0}{\sim} \quad \mathrm{Im}\mathcal{L}^{(1)}(E)\ \mathrm{e}^{\alpha\pi} \tag{1.17}$$

it states that an all-loop summation has produced the convergent factor $\mathrm{e}^{\alpha\pi}$, clearly an observation similar in vein to Cvitanovic's. Moreover, at a formal level it is not difficult to transfer this loop summation factor from $\mathrm{Im}\mathcal{L}(E)$ to the QED photon amplitudes [12, 14]. Consider the weak field expansion of the l-loop contribution to the electric EHL:

$$\mathcal{L}^{(l)}(E) \;=\; \sum_{n=2}^{\infty} c^{(l)}(n)\Bigl(\frac{eE}{m^2}\Bigr)^{2n} \tag{1.18}$$

Using the Borel dispersion relations discussed above, (1.15) can be shown [22, 12] to imply that, at any loop order l, the weak field expansion coefficients have the same leading asymptotic growth as we found above for the one-loop case, that is

$$c^{(l)}(n) \quad \overset{n\to\infty}{\sim} \quad c^{(l)}_{\infty}\, \pi^{-2n}\Gamma(2n-2) \tag{1.19}$$

where the constant $c^{(l)}_{\infty}$ relates directly to the prefactor of the corresponding leading Schwinger exponential in the weak field limit:

$$\mathrm{Im}\mathcal{L}^{(l)}(E) \quad \overset{E\to 0}{\sim} \quad c^{(l)}_{\infty}\ \mathrm{e}^{-\pi\frac{m^2}{eE}} \tag{1.20}$$

We would now like to convert (1.20) into an equation for photon amplitudes, using the above-mentioned correspondence between the weak field expansion coefficients and the EH limit of the photon amplitudes. Here we encounter the problem that this relation cannot be applied to a purely electric field; we

will therefore now switch to the SD case, and, on the strength of the two-loop results of [11, 12], assume that (1.17) holds for the SD case unchanged.

Now, the SD EHL relates to the 'all +' N - photon amplitudes, and for those the whole kinematic structure in the EH limit can, independently of the loop order, be absorbed into the invariant χ_N^+ defined in (1.3) [13]. One can thus eliminate this kinematic factor by dividing the l - loop amplitude by the one-loop one. Expanding (1.15) in α and combining it with (1.20) and (1.19) one then arrives at a formula for this ratio of amplitudes in the limit of large photon number [12],

$$\lim_{N\to\infty}\frac{\Gamma^{(l)}[k_1,\varepsilon_1^+;\dots;k_N,\varepsilon_N^+]}{\Gamma^{(1)}[k_1,\varepsilon_1^+;\dots;k_N,\varepsilon_N^+]} \quad = \quad \lim_{n\to\infty}\frac{c^{(l)}(n)}{c^{(1)}(n)} = \frac{(\alpha\pi)^{l-1}}{(l-1)!} \tag{1.21}$$

If we could now sum both sides over l and interchange the sum and limit, we could reconstruct the $\mathrm{e}^{\alpha\pi}$ factor, and conclude that the perturbation series for the N - photon amplitudes, if only for sufficiently large N, in the low energy limit and for this polarization component, has infinite convergence radius. But this is too good to be true, since so far we have nowhere made a distinction between quenched and unquenched contributions to the photon amplitudes, and convergence of the whole perturbation series can certainly be excluded.

However, as was noted in [14] this distinction comes in naturally if one takes into account that, as mentioned above, in the path integral derivation of (1.15) in [34] the rhs comes entirely from the quenched sector; all non-quenched contributions are suppressed in the weak field limit. And since (switching back to the usual Feynman diagram picture) the importance of non-quenched diagrams is growing with increasing loop order, it is natural to assume that their inclusion will slow down the convergence towards the asymptotic limit with increasing l, sufficiently to invalidate the above naive interchange of limits. On the other hand, there is no obvious reason to expect such a slowing down of convergence inside the quenched sector, which led to the prediction [14] that Cvitanovic's "quenched convergence" will indeed be found to hold true for the photon amplitudes.

To further corroborate this prediction, one should now calculate the EHL's and their weak field expansion coefficients $c^{(3)}(n)$ at the three-loop level. This would already allow one to perform some quite nontrivial checks on the above chain of reasoning, namely:

1. The asymptotic relation

$$\lim_{n\to\infty}\frac{c^{(3)}_{\rm scal,spin}(n)}{c^{(1)}_{\rm scal,spin}(n)} = \frac{(\alpha\pi)^2}{2} \tag{1.22}$$

and its independence of spin.

2. The absence of a slow-down of the convergence of the ratio (1.22) as compared to the corresponding two-loop to one-loop ratio.

3. The asymptotic subdominance of the non-quenched part of the EHL, which first appears at three loops.

However, a calculation of any three-loop EHL, be it in Scalar or Spinor QED, for an electric or self-dual field, presently still poses an enormous computational challenge.

Now, it is well-known that the structure of the EHL's and associated Schwinger exponentials at one-loop is essentially independent of the space-time dimension [36, 37, 38, 39]. In particular, the Schwinger formulas for 1+1 dimensional QED are different from (1.5), (1.6) only by normalization factors:

$$\begin{aligned}
{\rm Im}\mathcal{L}^{(1)(2D)}_{\rm spin}(E) &= \frac{eE}{4\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left[-\pi k\frac{m^2}{eE}\right] && (1.23)\\
{\rm Im}\mathcal{L}^{(1)(2D)}_{\rm scal}(E) &= -\frac{eE}{4\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\exp\left[-\pi k\frac{m^2}{eE}\right] && (1.24)
\end{aligned}$$

More surprisingly, in 2006 Dunne and Krasnansky [40, 41] found that, for the scalar QED case, even at the two-loop level there persists a strong similarity between the 2D EHL and the 4D self-dual EHL. For the former, they find

$$\mathcal{L}^{(2)(2D)}_{\rm scal}(\kappa) = -\frac{m^2}{2\pi}\frac{\tilde\alpha}{32}\left[\xi^2_{2D} - 4\kappa\xi'_{2D}\right] \tag{1.25}$$

where $\tilde\alpha := 2\frac{e^2}{\pi m^2}$ is our definition of the fine structure constant in two dimensions, and

$$\xi_{2D}(\kappa) \quad := \quad -\Bigl(\psi(\kappa+\frac{1}{2}) - \ln(\kappa)\Bigr) = \psi(\kappa) - 2\psi(2\kappa) + \ln(4\kappa) \tag{1.26}$$

Here the parameter f is defined by (see app. A for our 2D QED conventions)

$$F \quad = \quad \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix} \tag{1.27}$$

(note that this definition is consistent with (1.10)). The formal similarity between (1.25) and the 4D SD EHL (1.11) has led us to consider 2D QED as a toy model for studying the above asymptotic predictions. This will make sense only, of course, if the AAM formula (1.28) can be generalized to the 2D case. As we will show below, this generalization is

$$\mathrm{Im}\mathcal{L}_{\mathrm{scal}}^{(\mathrm{all-loop})(2D)}(E) \quad \overset{E\to 0}{\sim} \quad \frac{eE}{4\pi}\,\mathrm{e}^{-\frac{m^2\pi}{eE}+\tilde{\alpha}\pi^2\kappa^2} \tag{1.28}$$

As in the 4D case, we shall assume that spin does not play a role in this limit, so that (1.28) holds for spinor QED unchanged (let us also mention here that Loskutov et al. [42] have found an all-loop exponentiation formula similar to (1.28) for the strong-field asymptotics of the electron mass operator in a two-dimensional approximation of four-dimensional QED).

In the same way as in the 4D case, one can then use Borel analysis to derive from (1.28) a formula for the limits of ratios of l - loop to one - loop coefficients:

$$\lim_{n\to\infty}\frac{c_{2D}^{(l)}(n)}{c_{2D}^{(1)}(n+l-1)} \quad = \quad \frac{(\tilde{\alpha}\pi^2)^{l-1}}{(l-1)!} \tag{1.29}$$

Here our definition of the expansion coefficients in 2D is

$$\mathcal{L}^{(l)(2D)}(\kappa) \quad = \quad \frac{m^2}{2\pi}\sum_{n=1}^{\infty}(-1)^{l-1}c_{2D}^{(l)}(n)(i\kappa)^{-2n} \tag{1.30}$$

Note that, due to the factor of κ^2 appearing in the exponent on the rhs of (1.28), the 2D formula (1.29) involves also a shift in the argument of

$c^{(1)}_{2D}(n)$; the leading asymptotic growth of the coefficients increases with the loop order. This actually simplifies matters, since it implies that, for our asymptotic purposes, there is no need to perform mass renormalization: Mass renormalization contributions to an EHL at the l - loop level are of the form

$$\delta m^{(l_1)}\delta m^{(l_2)}\cdots\delta m^{(l_j)}\frac{\partial^j}{\partial m^j}\mathcal{L}^{(l')} \tag{1.31}$$

with some $l' < l$ and $l - l' = \sum_{i=1}^{j} l_i$, where $m^{(l_i)}$ denotes an l_i - loop mass counterterm. As we will show below, $c^{(1)}_{2D}(n)$ grows like $\Gamma(2n-1)$, and (1.29) then predicts that $c^{(l)}_{2D}(n)$ grows like $\Gamma(2n+2l-3)$. It is then easily seen that the expansion coefficients of a term of the structure (1.31) at l loops can grow at most like $\sim \Gamma(2n+2l-4)$, and thus are subdominant.

This is very different from the 4D case. Here the leading asymptotic growth of the expansion coefficients of the unrenormalized electric EHL's $\mathcal{L}^{(l)(4D)}_{\text{scal,spin}}$ is like $\Gamma(2n+l-3)$ ($\Gamma(2n+l-2)$ for the SD case), that is, the argument of the Γ - function jumps only by one per loop order [14]. Thus mass renormalization terms *do* contribute at the leading asymptotic level, and are even crucial to make the AAM formula work: At two loops, it has been shown [22, 12] that, precisely when the mass is taken to be the physical renormalized mass, the leading asymptotic terms cancel between the main contribution and the one from mass renormalization, which reduces the asymptotic growth at two loops to make it the same as at one-loop, as is implicit in the AAM formula (1.15). And for this formula to hold, these cancellations between main and counterterms must not only persist, but become increasingly extensive at higher loop orders. Thus the AAM formula already predicts that the asymptotic behaviour of QED must depend crucially on whether mass renormalization is done physically or just generically.

In section 2 we will first retrace the original derivation of the AAM formula and its generalization to spinor QED [43], and then present our generalization to the 2D QED case. In sections 3 and 4 we compute the one- and two- loop EHL's in 2D spinor QED. In section 5 we use these results, together with the formulas previously obtained in [40, 41] for scalar QED, to verify that the asymptotic predictions derived from our generalized AAM formula hold at two loop order in both theories. Section 6 gives a summary. The second part of this series [44] will be devoted to the calculation of the three-loop EHL's in scalar and spinor 2D QED, and to the verification of the 2D analogue of the three-loop predictions discussed above.

Some preliminary results of this work have been presented in [45].

2 Predictions of the AAM formalism

In this chapter, we will first retrace the derivation of the AAM formula (1.15), and then use their method to obtain also the generalization of this formula to the 2D case, eq. (1.28).

The seminal work of those authors concerned scalar QED in four dimensions, therefore let us start with this case, beginning at the one-loop level. The (euclidean) one-loop effective action for scalar QED can be written in the following way [35]:

$$
\begin{aligned}
\Gamma_{\rm scal}[A] &= \int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int_{x(T)=x(0)}\mathcal{D}x\;\mathrm{e}^{-S[x(\tau)]}\\
S[x(\tau)] &= \int_0^T d\tau\left(\frac{\dot x^2}{4}+ieA\cdot\dot x\right)
\end{aligned}
\tag{2.1}
$$

Here m is the mass of the scalar particle, and the functional integral $\int\mathcal{D}x$ is over all closed spacetime paths $x^\mu(\tau)$ which are periodic in the proper-time parameter τ, with period T. Rescaling $\tau = Tu$, the effective action may be expressed as

$$
\begin{aligned}
&\Gamma_{\rm scal}[A] = \\
&\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int_{x(1)=x(0)}\mathcal{D}x\,\exp\left[-\left(\frac{1}{4T}\int_0^1 du\,\dot x^2+ie\int_0^1 du\,A\cdot\dot x\right)\right]
\end{aligned}
\tag{2.2}
$$

After this rescaling we can perform the proper-time integral using the method of steepest descent. The T integral has a stationary point at

$$
T_0=\frac{1}{2m}\sqrt{\int_0^1 du\,\dot x^2}
\tag{2.3}
$$

leading to

$$
\mathrm{Im}\,\Gamma_{\rm scal}=\frac{1}{m}\sqrt{\frac{\pi}{T_0}}\,\mathrm{Im}\int\mathcal{D}x\;\mathrm{e}^{-\left(m\sqrt{\int\dot x^2}+ie\int_0^1 du A\cdot\dot x\right)}
\tag{2.4}
$$

Here we have implicitly used the large mass approximation

$$
m\sqrt{\int_0^1 du\,\dot x^2}\gg 1
\tag{2.5}
$$

The functional integral remaining in the effective action expression (2.4) may be approximated by a further, functional, stationary phase approximation. The new worldline "action",

$$S_{\rm eff} = m\sqrt{\int_0^1 du\, \dot{x}^2} + ie\int_0^1 du A\cdot\dot{x} \tag{2.6}$$

is stationary if the path $x_\alpha(u)$ satisfies

$$m\frac{\ddot{x}_\mu}{\sqrt{\int_0^1 du\, \dot{x}^2}} \;\; = \;\; ieF_{\mu\nu}\dot{x}_\nu \tag{2.7}$$

A periodic solution $x_\mu(u)$ to (2.7) is called a "worldline instanton". Contracting (2.7) with $\dot{x}_\mu$ shows that for such an instanton

$$\dot{x}^2 = \text{constant} \equiv a^2 \tag{2.8}$$

Generally, the existence of a worldline instanton for a background A leads to an imaginary part in the effective action $\Gamma_{\rm scal}[A]$, and the leading behavior is

$$\mathrm{Im}\Gamma_{\rm scal}[A] \sim e^{-S_0} \tag{2.9}$$

where S_0 is the worldline action (2.6) evaluated on the worldline instanton.

For a constant electric background of magnitude E, pointing in the z direction, the Euclidean gauge field is $A_3(x_4) = -iEx_4$. The instanton equation (2.7) for this case can be easily solved, and the solutions are simply circles in the $z-t$ plane of radius $\frac{m}{eE}$ [34]:

$$x_k^3(u) = \frac{m}{eE}\cos(2k\pi u) \quad , \quad x_k^4(u) = \frac{m}{eE}\sin(2k\pi u) \tag{2.10}$$

(with $x_{1,2}$ kept constant). The integer $k\in\mathbf{Z}^+$ counts the number of times the closed path is traversed, and the instanton action (2.6) becomes

$$S_0 := S_{\rm eff}[x_k^\mu] \;\; = \;\; 2k\,\frac{m^2\pi}{eE} - k\,\frac{m^2\pi}{eE} = k\,\frac{m^2\pi}{eE} \tag{2.11}$$

Thus in the large mass approximation (2.5) the contribution of the instanton with winding number k reproduces the exponent of the kth term of Schwinger's formula (1.5).

Proceeding to the multiloop case, Feynman's formula (2.1) can be easily modified to take into account the effect of multiple exchanges of photons in the scalar loop. This requires only the addition of the following interaction term S_i to the worldline action,

$$S_i \quad = \quad \frac{e^2}{8\pi^2}\int_0^T d\tau_1 \int_0^T d\tau_2 \frac{\dot{x}(\tau_1)\cdot\dot{x}(\tau_2)}{(x(\tau_1)-x(\tau_2))^2} \tag{2.12}$$

After this addition in (2.1), $\Gamma_{\rm scal}[A]$ turns into the "quenched" effective action, corresponding to the set of all Feynman diagrams with a single scalar loop (but any number of interactions with the external field and internal photon exchanges). Now, Affleck et al. [34] argue that the instanton solutions (2.10) remain stationary even in the presence of this addition, so that its effect in the large mass limit is only a modification of the stationary action. Evaluation of S_i on the leading $k=1$ instanton yields simply

$$S_i[x(u)] \quad = \quad -\alpha\pi \tag{2.13}$$

Combining (2.9),(2.11),(2.13) one obtains the large mass approximation for the quenched multiloop scalar EHL (1.15),

$$\mathrm{Im}\mathcal{L}_{\rm scal}^{(\rm all-loop)}(E) \quad \stackrel{E\to 0}{\sim} \quad \frac{(eE)^2}{16\pi^3}\exp\Big[-\pi\frac{m^2}{eE}+\alpha\pi\Big]$$

Affleck et al. then proceed to show that the contributions to the full effective action involving more than one scalar loop are suppressed with respect to the quenched one in this large mass limit, so that (1.15) actually holds even for the full effective action. Even more remarkably, they argue that the asymptotic formula (1.15) already takes into account all the effects of mass renormalization which come into play for this effective action starting with the two-loop level.

The approach of Affleck et al. was generalized in [43] to the case of spinor QED. The path integral representation of the one-loop effective action due to a spin half particle in the loop differs from (2.1) only by a global factor of $-\frac{1}{2}$ and the insertion of the following 'spin factor' $S[x,A]$ under the path integral [46],

$$S[x,A] = \mathrm{tr}_{\Gamma}\mathcal{P}\ \mathrm{e}^{\frac{i}{2}e\sigma^{\mu\nu}\int_0^T d\tau F_{\mu\nu}(x(\tau))} \tag{2.14}$$

Here tr_Γ denotes the Dirac trace and $\mathcal{P}$ the path ordering operator. The presence of this spin factor term in the stationary path approximation is again taken into account simply by evaluating this term on the stationary path (2.10). The result turns out to be a simple global factor [43],

$$4\cos(\pi n) = 4(-1)^n \tag{2.15}$$

Taking the global factor of $-\frac{1}{2}$ into account this correctly reproduces the difference between the one-loop Schwinger formulas for scalar and spinor QED (1.5),(1.6). Proceeding to spinor QED at the multiloop level, here there seems to be no simple way to directly calculate the effect of the spin term in the stationary path approximation. However, as was discussed already in the introduction the physical arguments of [32] as well as explicit two-loop calculations strongly suggest that the AAM formula (1.15) holds also for the spinor QED case (up to a global factor of 1/2).

We will now generalize the AAM formula to the case of QED in two dimensions. First, we note that the path integral representation (2.1) for the scalar QED effective action is independent of dimension; the only explicit appearance of the dimension D is in the normalization of the free path integral, which is

$$\int_{x(T)=x(0)} \mathcal{D}x \exp\left[-\int_0^T d\tau \, \frac{\dot{x}^2}{4}\right] \quad = \quad (4\pi T)^{-D/2} \tag{2.16}$$

(with the conventions of [29]). This does not affect any of the manipulations which we performed at the one-loop level, including the form of the worldline instanton (2.10), so that the asymptotic estimate (2.9) remains valid in $D = 2$ with the same formula (2.11) for S_0.

Things are different at the multiloop level. Here it must be remembered that the worldline insertion term (2.12) actually involves the photon propagator in $D = 4$ and in Feynman gauge. For general D and in a general covariant gauge this term reads

$$\begin{aligned} S_i(D) \quad &= \quad \frac{e^2}{2}\frac{1}{4\pi^{\frac{D}{2}}}\int_0^T d\tau_a \int_0^T d\tau_b \biggl\{ \frac{1+\alpha}{2}\Gamma\Bigl(\frac{D}{2}-1\Bigr)\frac{\dot{x}_a\cdot\dot{x}_b}{\Bigl[(x_a-x_b)^2\Bigr]^{\frac{D}{2}-1}} \\ &\quad +(1-\alpha)\Gamma\Bigl(\frac{D}{2}\Bigr)\frac{\dot{x}_a\cdot(x_a-x_b)(x_a-x_b)\cdot\dot{x}_b}{\Bigl[(x_a-x_b)^2\Bigr]^{\frac{D}{2}}}\biggr\} \end{aligned} \tag{2.17}$$

where $\alpha = 1$ corresponds to Feynman gauge. Since the first term in braces in (2.17) becomes singular in $D = 2$, in this case instead of Feynman gauge it is more convenient to choose the gauge $\alpha = -1$, leading to

$$S_i^{D=2} \quad = \quad \frac{e^2}{4\pi}\int_0^T d\tau_a \int_0^T d\tau_b \frac{\dot{x}_a \cdot (x_a - x_b)(x_a - x_b)\cdot \dot{x}_b}{(x_a - x_b)^2} \tag{2.18}$$

The evaluation of $S_i^{D=2}$ on the constant field worldline instanton (2.10) then yields

$$S_i^{D=2}[x(u)] \quad = \quad \frac{\pi}{2}\frac{m^2}{E^2} = 2\pi(i\kappa)^2\frac{e^2}{m^2} = -\tilde{\alpha}\pi^2\kappa^2 \tag{2.19}$$

This brings us to our 2D generalization of the AAM formula, eq. (1.28).

3 1 loop Euler-Heisenberg Lagrangian

The (renormalized) one-loop Euler-Heisenberg Lagrangians in 2D scalar and spinor QED are, in the standard proper-time representation (see, e.g., [36]),

$$\mathcal{L}_{\rm scal}^{(1)}(f) \quad = \quad \frac{ef}{4\pi}\int_0^\infty \frac{dz}{z}\,\mathrm{e}^{-2\kappa z}\Bigl(\frac{1}{\sinh(z)} - \frac{1}{z}\Bigr) \tag{3.1}$$

$$\mathcal{L}_{\rm spin}^{(1)}(f) \quad = \quad -\frac{ef}{4\pi}\int_0^\infty \frac{dz}{z}\,\mathrm{e}^{-2\kappa z}\Bigl(\coth(\mathrm{z}) - \frac{1}{\mathrm{z}}\Bigr) \tag{3.2}$$

It is convenient to observe that the scalar EHL can be written in terms of the spinor EHL as [47]

$$\mathcal{L}_{\rm scal}^{(1)}(f) = \mathcal{L}_{\rm spin}^{(1)}(f) - 2\mathcal{L}_{\rm spin}^{(1)}(f/2) \tag{3.3}$$

The integrals (3.1),(3.2) have various equivalent closed-form representations; the most suitable one for our purposes is

$$\mathcal{L}_{\rm spin}^{(1)}(\kappa) \quad = \quad -\frac{m^2}{4\pi}\frac{1}{\kappa}\Bigl[\ln\Gamma(\kappa) - \kappa(\ln\kappa - 1) + \frac{1}{2}\ln\bigl(\frac{\kappa}{2\pi}\bigr)\Bigr] \tag{3.4}$$

4 2 loop Euler-Heisenberg Lagrangian

We now calculate the two-loop Euler-Heisenberg Lagrangian. In momentum space, it is given by

$$\mathcal{L}^{(2)}_{\rm spin}(f) = \frac{e^2}{2}\int\frac{d^2p}{(2\pi)^2}\int\frac{d^2p'}{(2\pi)^2}\,{\rm tr}\Bigl[g(p)\sigma_\mu g(p')\sigma_\mu\Bigr]\frac{1}{(p-p')^2} \tag{4.1}$$

After plugging in (A.8), we note that due to (A.10) the terms involving $\sigma\cdot p$ do not contribute under the Dirac trace, which thus becomes simply

$${\rm tr}\left[\,{\rm e}^{\sigma^3 z}\sigma_\mu\,{\rm e}^{\sigma^3 z'}\sigma_\mu\right] = {\rm tr}\left[\,{\rm e}^{\sigma^3 z}\,{\rm e}^{-\sigma^3 z'}\sigma_\mu\sigma_\mu\right] = 4\cosh(z-z') \tag{4.2}$$

This leaves us with

$$\begin{aligned}\mathcal{L}^{(2)}_{\rm spin}(f) &= 2e^2m^2\int_0^\infty dT\int_0^\infty dT'\,{\rm e}^{-m^2(T+T')}\frac{\cosh(z-z')}{\cosh z\cosh z'}\\ &\quad\times\int\frac{d^2p}{(2\pi)^2}\int\frac{d^2p'}{(2\pi)^2}\frac{1}{(p-p')^2}\,{\rm e}^{-\frac{1}{ef}({\rm tanhz\,p^2+tanhz'\,p'^2})}\end{aligned} \tag{4.3}$$

We exponentiate the photon propagator,

$$\frac{1}{(p-p')^2} = \int_0^\infty d\lambda\,{\rm e}^{-\lambda(p-p')^2} \tag{4.4}$$

The momentum integrals can now be done, yielding

$$\int d^2p\int d^2p'\,{\rm e}^{-\lambda(p-p')^2-\frac{1}{ef}({\rm tanhz\,p^2+tanhz'\,p'^2})} = \frac{\pi^2(ef)^2}{{\rm tanhztanhz'}+\lambda\,{\rm ef}({\rm tanhz}+{\rm tanhz'})} \tag{4.5}$$

The λ - integral has a logarithmic infrared divergence which we regulate with a cutoff λ_0, $\int_0^\infty d\lambda\to\int_0^{\lambda_0}d\lambda$. Using

$$\frac{1}{\cosh z\cosh z'({\rm tanhz}+{\rm tanhz'})} = \frac{1}{\sinh(z+z')} \tag{4.6}$$

the result becomes, in the limit $\lambda_0 ef\gg 1$,

$$\mathcal{L}^{(2)}_{\rm spin}(f) = \frac{e^3 f m^2}{8\pi^2}\int_0^\infty dT \int_0^\infty dT' \, {\rm e}^{-m^2(T+T')}\frac{\cosh(z-z')}{\sinh(z+z')}\Bigl[\ln(ef\lambda_0)+\ln\Bigl(\frac{\sinh(z+z')}{\sinh z \sinh z'}\Bigr)\Bigr] \tag{4.7}$$

For the calculation of the integrals, it is convenient to change to the "worldline" variables [29],

$$\begin{aligned} T' &= u\tilde{T} \\ T &= (1-u)\tilde{T} \\ \int_0^\infty dT \int_0^\infty dT' &= \int_0^\infty d\tilde{T}\tilde{T}\int_0^1 du \end{aligned} \tag{4.8}$$

There are two different u integrals, both elementary:

$$\begin{aligned} \int_0^1 du \cosh\bigl[(1-2u)Z\bigr] &= \frac{\sinh(Z)}{Z} \\ \int_0^1 du \cosh\bigl[(1-2u)Z\bigr]\ln\sinh(uZ) &= -\frac{1}{2}\cosh Z + \frac{\sinh Z}{Z}\bigl(\ln\sinh Z - \frac{1}{2}\bigr) \end{aligned} \tag{4.9}$$

where $Z \equiv ef\tilde{T}$. After some rearrangements, the result can be written as

$$\begin{aligned} \mathcal{L}^{(2)}_{\rm spin}(f) &= \frac{m^2e^2}{8\pi^2}\frac{1}{ef}\int_0^\infty dZ\, {\rm e}^{-2\kappa Z}\Bigl[Z({\rm cothZ}-\frac{1}{\rm Z}) - \ln\sinh{\rm Z} + \ln{\rm Z}\Bigr] \\ &\quad + \frac{e^2}{8\pi^2}\Bigl[\ln(\lambda_0 m^2)+2+\gamma\Bigr] \end{aligned} \tag{4.10}$$

(γ is the Euler-Mascheroni constant). The remaining integrals can be easily reduced to the standard integral [48]

$$\int_0^\infty dZ\, {\rm e}^{-2\kappa Z}({\rm cothZ}-\frac{1}{\rm Z}) = \frac{\xi(\kappa)}{\kappa} \tag{4.11}$$

Putting the pieces together, our final result for the (unrenormalized) 2-loop spinor EHL in 2D is

$$\mathcal{L}^{(2)}_{\rm spin}(\kappa) \;=\; \frac{m^2}{4\pi}\frac{\tilde{\alpha}}{4}\Bigl[-\xi'(\kappa)+\ln(\lambda_0 m^2)+\gamma+2\Bigr] \tag{4.12}$$

Thus, contrary to the scalar QED case, the two-loop 2D spinor QED EHL turns out to be significantly simpler than the corresponding self – dual 4D one (1.12). In fact, it has the curious property that, up to the vacuum part, one can even write it in terms of derivatives of the one-loop EHL (3.4):

$$\mathcal{L}^{(2)}_{\rm spin}(\kappa) \;=\; -\frac{\tilde{\alpha}}{4}\Bigl(m^2\frac{\partial}{\partial m^2}\Bigr)^2\mathcal{L}^{(1)}_{\rm spin}(\kappa) \tag{4.13}$$

Note that, as explained in the introduction, there is no need for us to perform the mass renormalization of $\mathcal{L}^{(2)}_{\rm spin}$.

5 Weak field expansions

Finally, we will now work out the weak field expansions of the one and two-loop $2D$ EHL's, for both the scalar and spinor cases, and verify the asymptotic prediction (1.29) derived from the AAM formalism.

Starting with the spinor case, for working out the one-loop EHL (3.4) we need the large x expansion of $\ln\Gamma(x)$, which is

$$\ln\Gamma(x)\sim x(\ln x-1)-\frac{1}{2}\ln\Bigl(\frac{x}{2\pi}\Bigr)+\sum_{n=1}^{\infty}\frac{B_{2n}}{2n(2n-1)}x^{-(2n-1)} \tag{5.1}$$

This yields (the subscript '2D' will be omitted in this chapter)

$$c^{(1)}_{\rm spin}(n) \;=\; (-1)^{n+1}\frac{B_{2n}}{4n(2n-1)} \tag{5.2}$$

($n\geq 1$). At the two-loop level, we get from (5.1) that

$$-\,\xi'(x) \;\sim\; \sum_{n=1}^{\infty}\frac{2n-1}{2n}B_{2n}x^{-2n} \tag{5.3}$$

Thus (4.12) gives (omitting the cutoff-dependent vacuum energy term)

$$c^{(2)}_{\rm spin}(n) \quad = \quad (-1)^{n+1}\frac{\tilde\alpha}{8}\frac{2n-1}{2n}B_{2n} \tag{5.4}$$

$(n \geq 1)$. Using Euler's formula

$$B_{2n} \quad = \quad (-1)^{n+1}2(2\pi)^{-2n}(2n)!\zeta(2n) \tag{5.5}$$

and $\lim_{n\to\infty}\zeta(n) = 1$ we find

$$\begin{aligned} c^{(1)}_{\rm spin}(n) \quad &\sim \quad \frac{\Gamma(2n-1)}{(2\pi)^{2n}} \\ c^{(2)}_{\rm spin}(n) \quad &\sim \quad \frac{\tilde\alpha}{4}\frac{\Gamma(2n+1)}{(2\pi)^{2n}} \end{aligned} \tag{5.6}$$

so that

$$\lim_{n\to\infty}\frac{c^{(2)}_{\rm spin}(n)}{c^{(1)}_{\rm spin}(n+1)} \quad = \quad \tilde\alpha\pi^2 \tag{5.7}$$

in agreement with (1.29).

Proceeding to the scalar QED case, at one-loop we get, from (3.3) and (5.2),

$$c^{(1)}_{\rm scal}(n) \quad = \quad (-1)^{n+1}(1-2^{1-2n})\frac{B_{2n}}{4n(2n-1)} \tag{5.8}$$

$(n \geq 1)$. Comparing with (5.2), we see that $c^{(1)}_{\rm scal}(n) \sim c^{(1)}_{\rm spin}(n)$ for large n. At two loops, we need the expansion of the function $\xi_{2D}(\kappa)$, defined in (1.26), which is

$$\xi_{2D}(\kappa) = \sum_{n=1}^{\infty}\frac{\bar B_{2n}}{2n}\kappa^{-2n} \tag{5.9}$$

where we have defined $\bar B_n := (2^{1-n}-1)B_n$. The expansion of (1.25) then gives

$$c^{(2)}_{\rm scal}(n) = \frac{\tilde\alpha}{8}(-1)^n\Bigl(\sum_{m=1}^{n-1}\frac{\bar B_{2m}}{4m}\frac{\bar B_{2n-2m}}{4(n-m)} + \bar B_{2n}\Bigr) \tag{5.10}$$

($n \geq 2$). The structure of these coefficients is thus very similar to the one of the expansion coefficients of the two-loop EHL's for a self-dual field in 4D [11, 12]. In [12] also a method was developed to compute the asymptotic expansion of folded sums of Bernoulli numbers of the type appearing in the first term in brackets in (5.10); applying the same technique to the case at hand one can show that this term is subdominant in the large n limit. The leading order contribution comes from the second term, and yields

$$c^{(2)}_{\rm scal}(n) \quad \sim \quad \frac{\tilde\alpha}{4}\frac{\Gamma(2n+1)}{(2\pi)^{2n}} \tag{5.11}$$

Comparing with (5.4), we see that the two-loop coefficients $c^{(2)}_{\rm scal}(n)$ and $c^{(2)}_{\rm spin}(n)$ also agree for large n. This completes our check of (5.7) for the scalar case.

6 Conclusions

To summarize, in the first part of this series we have, generalizing the work of Affleck et al., obtained an all-loop formula for the imaginary part of the Euler-Heisenberg lagrangian in 1+1 QED, in the limit of a weak field but at arbitrary coupling. We have performed the first calculation of the Euler-Heisenberg lagrangian in 2D spinor QED at two loops, and verified that the asymptotic behaviour of its weak field expansion coefficients agrees with the prediction of this generalized AAM formula. We have also performed the same check for the scalar QED case, using the two-loop result of Dunne and Krasnansky. Our findings clearly demonstrate that 2D QED is sufficiently close to the 4D case to suggest that it may hold generic information on the asymptotic behaviour of amplitudes in QED in general, but at the same time holds more promise for the explicit study of multiloop Euler-Heisenberg lagrangians. A particularly interesting common aspect of our two-loop 2D QED results and the two-loop SD EHL calculations of [11, 12] is, that, in all cases where an explicit formula has been obtained for a two-loop EHL, the leading asymptotic growth of its expansion coefficients has turned out

to involve only the digamma function, and only linearly. We take this as further evidence that the study of the N-photon amplitudes through the QED effective Lagrangian may ultimately provide a window to high orders in perturbation theory.

Acknowledgements: C. S. thanks G.V. Dunne and H. Gies for discussions. I. H. and C. S. thank CONACYT for financial support. D.G.C. M. thanks the IFM, UMSNH for hospitality during part of this work.

A Conventions and formulas for Euclidean 1+1 QED

Dirac equation:

$$\Bigl(\sigma_\mu(\partial_\mu - ieA_\mu) + m\Bigr)\psi \quad = \quad 0 \tag{A.1}$$

Free electron propagator:

$$\frac{1}{i\not{p} + m} \quad = \quad \frac{-i\not{p} + m}{p^2 + m^2} \tag{A.2}$$

($\not{p} = \sigma_\mu p_\mu$).

Photon propagator in Feynman gauge:

$$\frac{\delta_{\mu\nu}}{k^2} \tag{A.3}$$

Vertex:

$$ie\sigma_\mu \tag{A.4}$$

Constant field:

$$F \quad = \quad \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix} \tag{A.5}$$

Fock-Schwinger gauge:

$$A_\mu(x) = -\frac{1}{2}F_{\mu\nu}x_\nu \tag{A.6}$$

Electron propagator in a constant field in Fock-Schwinger gauge:

$$\big[\sigma_\mu(\partial_\mu - ieA_\mu) + m\big]g(x-x') = \delta(x-x') \tag{A.7}$$

$$g(p) = \int_0^\infty dT\, \mathrm{e}^{-T\left(m^2+\frac{\tanh z}{z}p^2\right)}\frac{1}{\cosh z}\Big(m\,\mathrm{e}^{\sigma^3 z} - \frac{i\not{p}}{\cosh z}\Big) \tag{A.8}$$

$$\begin{aligned} g(x) &= \int \frac{d^2p}{(2\pi)^2}\,\mathrm{e}^{ip\cdot x}g(p) \qquad (A.9)\\ &= \frac{1}{4\pi}\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\frac{z}{\sinh z}\,\mathrm{e}^{-\frac{z}{\tanh z}\frac{x^2}{4T}}\Big(m\,\mathrm{e}^{\sigma^3 z} + \frac{1}{2T}\frac{z}{\sinh z}\not{x}\Big)\end{aligned}$$

$(z = efT)$.

One of the motivations for considering the Euler-Heisenberg Lagrangian in $1+1$ dimensions is that substantive simplifications can be expected for higher loop calculations in Feynman gauge due to the fact that

$$\sigma_\mu\sigma_{\nu_1}\cdots\sigma_{\nu_{2n+1}}\sigma_\mu = 0 \tag{A.10}$$

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